MATH 512, SPRING 17
HOMEWORK 2, DUE WED FEBRUARY 22

Problem 1. *Recall that $Add(\kappa, \lambda)$ is the poset of all partial functions from $\lambda \times \kappa \to \{0, 1\}$, with domains of size less than κ. The ordering is reverse inclusion, i.e. $p \leq q$ is as functions $p \supset q$.*

1. *Show that for all $\alpha < \lambda, \eta < \kappa$, $D_{\alpha,\eta} := \{p \mid \langle \alpha, \eta\rangle \in \mathrm{dom}(p)\}$ is dense.*
2. *Show that for all distinct $\alpha, \beta < \lambda$, the set $D := \{p \mid (\exists \eta, \delta < \kappa)(\{\langle \alpha, \eta\rangle, \langle \beta, \delta\rangle\} \subset \mathrm{dom}(p),$ and $p(\langle \alpha, \eta\rangle) \neq p(\langle \beta, \delta\rangle)\}$ is dense.*
3. *Use the two items above to show that if G is $Add(\kappa, \lambda)$-generic, then in $V[G]$, there are λ many distinct subsets of κ.*

Problem 2. *Suppose that $\pi : \mathbb{P} \to \mathbb{Q}$ is such that:*

1. *if $p' \leq p$ in $\mathbb{P}$, then $\pi(p') \leq \pi(p)$, and*
2. *for all $p \in \mathbb{P}$ and $q \leq \pi(p)$, there is $p' \leq p$, such that $\pi(p') \leq q$.*

Show that if G is $\mathbb{P}$-generic, then the upwards closure of the image i.e. $\{q \in \mathbb{Q} \mid \exists p \in G(\pi(p) \leq q)\}$ is $\mathbb{Q}$-generic. Such a map π is called a projection.

Problem 3. *Suppose that μ is a regular cardinal, $\mathbb{P}$ has the μ-chain condition and $p \Vdash$ "$\dot{B}$ is a bounded μ". Show that for some $\alpha < \mu$, $p \Vdash \dot{B} \subset \alpha$.*

Hint: Consider a maximal antichain below p, subset of $\{q \leq p \mid (\exists \alpha) q \Vdash \dot{B} \subset \alpha\}$

Recall that $\langle C_\alpha \mid \alpha \in \mathrm{Lim}(\kappa^+)\rangle$ is a $\square_\kappa$ sequence iff:

1. each C_α is a club subset of α,
2. for each α, if $\mathrm{cf}(\alpha) < \kappa$, then $o.t.(C_\alpha) < \kappa$,
3. for each α, if $\beta \in \mathrm{Lim}(C_\alpha)$, then $C_\alpha \cap \beta = C_\beta$.

Problem 4. *Suppose that $\langle C_\alpha \mid \alpha \in Lim(\kappa^+)\rangle$ is a $\square_\kappa$ sequence. Show that there is no club $C \subset \kappa^+$ such that for all α, $C \cap \alpha = C_\alpha$.*
Hint: look at the order type of initial segments of such a C.

Problem 5. *Let $j : V \to M$ be an elementary embedding with critical point κ. Suppose that $\mathbb{P}$ is Prikry forcing at κ (for some measure) and let G be $\mathbb{P}$-generic. Recall that in $V[G]$, κ is a singular cardinals with cofinality ω. Show that we cannot lift the embedding j to $V[G]$. In particular, show that any for any generic filter H for $j(\mathbb{P})$, we cannot have that j"$G \subset H$.*